

BOTT PERIODICITY, SUBMANIFOLDS, AND VECTOR BUNDLES

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ABSTRACT. We sketch a geometric proof of the classical theorem of Atiyah, Bott, and Shapiro [3] which relates Clifford modules to vector bundles over spheres. Every module of the Clifford algebra Cl_k defines a particular vector bundle over \mathbb{S}^{k+1} , a generalized Hopf bundle, and the theorem asserts that this correspondence between Cl_k -modules and stable vector bundles over \mathbb{S}^{k+1} is an isomorphism modulo Cl_{k+1} -modules. We prove this theorem directly, based on explicit deformations as in Milnor's book on Morse theory [8], and without referring to the Bott periodicity theorem as in [3].

INTRODUCTION

Topology and Geometry are related in various ways. Often topological properties of a specific space are obtained by assembling its local curvature invariants, like in the Gauss-Bonnet theorem. Bott's periodicity theorem is different: A detailed investigation of certain totally geodesic submanifolds in specific symmetric spaces leads to fundamental insight not just for these spaces but for whole areas of mathematics. This geometric approach was used originally by Bott [4, 5] and Milnor in his book on Morse theory [8] where the stable homotopy of the classical groups was computed. Later Bott's periodicity theorem was re-interpreted as a theorem on K-theory [2, 3, 1], but the proofs were different and less geometric. However we feel that the original approach of Bott and Milnor can prove also the K-theoretic versions of the periodicity theorem. As an example we discuss Theorem (11.5.) from the fundamental paper [3] by Atiyah, Bott and Shapiro, which relates Clifford modules to vector bundles over spheres. The argument in [3] uses explicit computations of the right and left hand sides of the stated isomorphism, and depends on the Bott periodicity theorem for the orthogonal groups. Instead we prove bijectivity of the relevant

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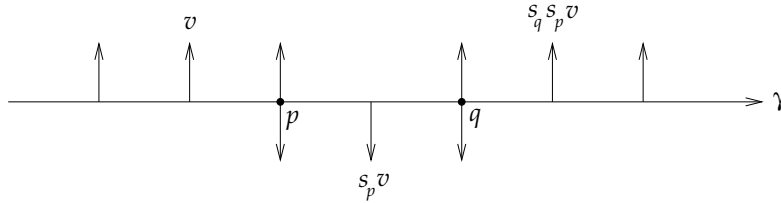
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comparison map directly. In consequence the Bott periodicity theorem for the orthogonal groups is now implied by its algebraic counterpart in the representation theory of Clifford algebras [3]. This gives a positive response to the remark in [3, page 4]: “It is to be hoped that Theorem (11.5) can be give a more natural and less computational proof”, cf. also [7, page 69]. We will concentrate on the real case which is more interesting and less well known than the complex theory. Much of the necessary geometry was explained to us by Peter Quast [12].

1. POLES AND CENTRIOLES

We start with the geometry. A *symmetric space* is a Riemannian manifold P with an isometric point reflection s_p (called *symmetry*) at any point $p \in P$, that is $s_p \in \hat{G} =$ isometry group of P with $s_p(\exp_p(v)) = \exp_p(-v)$ for all $v \in T_p P$. The map $s : p \mapsto s_p : P \rightarrow \hat{G}$ is called *Cartan map*; it is a covering onto its image $s(P) \subset \hat{G}$ which is also symmetric.¹ The composition of any two symmetries, $\tau = s_q s_p$ is called a *transvection*. It translates the geodesic γ connecting $p = \gamma(0)$ to $q = \gamma(r)$ by $2r$ and acts by parallel translation along γ , see next figure. The subgroup of \hat{G} generated by all transvections (acting transitively on P) will be called G .



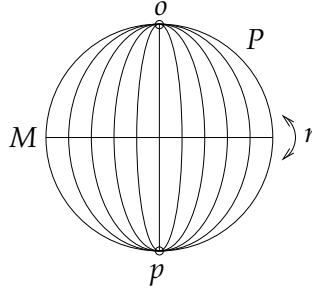
Two points $o, p \in P$ will be called *poles* if $s_p = s_o$. The notion was coined for the north and south pole of a round sphere, but there are many other spaces with poles; e.g. $P = SO_{2n}$ with $o = I$ and $p = -I$, or the Grassmannian $P = \mathbb{G}_n(\mathbb{R}^{2n})$ with $o = \mathbb{R}^n$ and $p = (\mathbb{R}^n)^\perp$. A geodesic γ connecting $o = \gamma(0)$ to $p = \gamma(1)$ is reflected into itself at o and p and hence it is closed with period 2.

Now we consider the *midpoint set* M between poles o and p ,

$$M = \{m = \gamma\left(\frac{1}{2}\right) : \gamma \text{ geodesic in } P \text{ with } \gamma(0) = o, \gamma(1) = p\}.$$

For the sphere $P = \mathbb{S}^n$ with north pole o , this set would be the equator, see figure below.

¹ $s(P) \subset \hat{G}$ is a connected component of the set $\{g \in \hat{G} : g^{-1} = g\}$. When we choose a symmetric metric on \hat{G} such that $g \mapsto g^{-1}$ is an isometry, $s(P)$ is a reflective submanifold and hence totally geodesic, thus symmetric.



Theorem 1. [11] *M is the fixed set of an isometric involution r on P.*

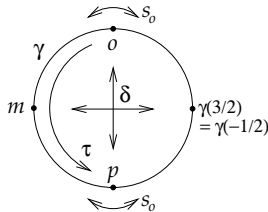
Proof. In the example of the sphere $P = \mathbb{S}^n$, the equator M is the fixed set of $-s_o = -I \circ s_o$. Here, $-I$ is the deck transformation² of the covering $\mathbb{S}^n \rightarrow \mathbb{RP}^n = \mathbb{S}^n / \{\pm I\}$. In the general case we consider the covering $P \rightarrow s(P)$. Since $s(P)$ is again symmetric, we have $s(P) = P/\Delta$ for some discrete freely acting group $\Delta \subset \hat{G}$ normalized by all symmetries and centralized by all transvections.³ Since $s_o = s_p$, the points o and p are identified in $s(P)$. Thus there is a unique $\delta \in \Delta$ with $\delta(o) = p$. This will be the analogue of $-I$ in the case $P = \mathbb{S}^n$. We will show that δ has order 2 and preserves any geodesic γ with $\gamma(0) = o$ and $\gamma(1) = p$. In fact, let τ be the transvection along γ from o to p . Then $\tau^2(o) = o$ and therefore

$$\delta(p) = \delta(\tau(o)) = \tau(\delta(o)) = \tau(p) = o.$$

Thus δ^2 fixes o which shows $\delta^2 = \text{id}$ since Δ acts freely. Hence $\{I, \delta\} \subset \Delta$ is a subgroup and $\bar{P} = P/\{\text{id}, \delta\}$ a symmetric space. Under the projection $\pi : P \rightarrow \bar{P}$, the geodesic γ is mapped onto a closed geodesic doubly covered by γ , thus δ preserves γ and shifts its parameter by 1, and γ has period 2.

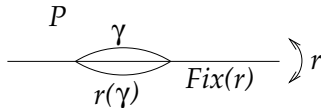
²A deck transformation of $\pi : P \rightarrow \bar{P}$ is an isometry δ of P with $\pi \circ \delta = \pi$.

³Consider a symmetric space P and a covering $\pi : P \rightarrow P/\Delta$ for some discrete freely acting group Δ of isometries on P . Then P/Δ is again symmetric if and only if each symmetry s_p of P maps Δ -orbits onto Δ -orbits. Thus for each $\delta \in \Delta$ we have $s_p(\delta x) = \tilde{\delta}s_p(x)$ for all $x \in P$, and $\tilde{\delta} \in \Delta$ is independent of x , by discreteness. Thus $s_p\delta = \tilde{\delta}s_p$, in particular $s_p\delta s_p = \tilde{\delta} \in \Delta$. For any other symmetry s_q we have the same equation $s_q\delta = \tilde{\delta}s_q$ with the same $\tilde{\delta} \in \Delta$, again by discreteness. Thus $\delta^{-1}s_p s_q \delta = s_p \tilde{\delta}^{-1} \tilde{\delta} s_q = s_p s_q$, and δ commutes with the transvection $s_p s_q$ (see also [14, Thm. 8.3.11]).

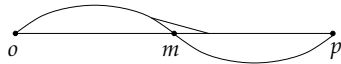


Vice versa, assume that $m \in P$ is a fixed point of r . Thus $s_o m = \delta m$. Join o to m by a geodesic γ with $\gamma(0) = o$ and $\gamma(\frac{1}{2}) = m$. Then $\gamma(-\frac{1}{2}) = s_o(m) = \delta(m) = \delta(\gamma(\frac{1}{2}))$, and the projection $\pi : P \rightarrow \bar{P} = P/\{\text{id}, \delta\}$ maps $\gamma : [-\frac{1}{2}, \frac{1}{2}] \rightarrow P$ onto a geodesic loop $\bar{\gamma} = \pi \circ \gamma$, that is a closed geodesic of period 1 (since \bar{P} is symmetric). Thus γ extends to a closed geodesic of period 2 doubly covering $\bar{\gamma}$, and δ shifts the parameter of γ by 1. Therefore $\gamma(1) = \delta(o) = p$. Hence m is the midpoint of $\gamma|_{[0,1]}$ from o to p . Thus $M \supset \text{Fix}(r)$. \square

Connected components of the midpoint set M are called *centrioles* [6]. Connected components of the fixed set of an isometry are totally geodesic (otherwise shortest geodesic segments in the ambient space with end points in the fixed set were not unique, see figure below); if the isometry is an involution, its fixed components are called *reflective*.



Most interesting are connected components containing midpoints of geodesics with *minimal* length between o and p (“*minimal centrioles*”). Each such midpoint $m = \gamma(\frac{1}{2})$ determines its geodesic γ uniquely: if there were two geodesics of equal length from o to p through m , they could be made shorter by cutting the corner.



There exist chains of minimal centrioles (centrioles in centrioles),

$$(1) \quad P \supset P_1 \supset P_2 \supset \dots$$

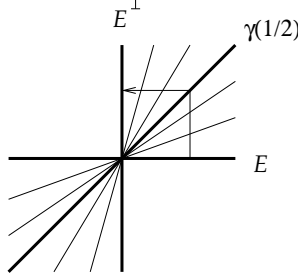
Peter Quast [12, 13] classified all such chains with at least 3 steps starting with a compact simple Lie group $P = G$. Up to group coverings,

the result is as follows. The chains 1,2,3 occur in Milnor [8].

	No.	G	P_1	P_2	P_3	P_4	restr.
(2)	1	$(S)O_{4n}$	SO_{4n}/U_{4n}	U_{2n}/Sp_n	$\mathbb{G}_p(\mathbb{H}^n)$	Sp_p	$p = \frac{n}{2}$
	2	$(S)U_{2n}$	$\mathbb{G}_n(\mathbb{C}^{2n})$	U_n	$\mathbb{G}_p(\mathbb{C}^n)$	U_p	$p = \frac{n}{2}$
	3	Sp_n	Sp_n/U_n	U_n/SO_n	$\mathbb{G}_p(\mathbb{R}^n)$	SO_p	$p = \frac{n}{2}$
	4	$Spin_n$	Q_n	$(\mathbb{S}^1 \times \mathbb{S}^{n-3})/\pm$	\mathbb{S}^{n-4}	\mathbb{S}^{n-5}	$n \geq 5$
	5	E_7	$E_7/(\mathbb{S}^1 E_6)$	$\mathbb{S}^1 E_6/F_4$	$\mathbb{O}\mathbb{P}^2$	—	

By $\mathbb{G}_p(\mathbb{K}^n)$ we denote the Grassmannian of p -dimensional subspaces in \mathbb{K}^n for $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$. Further, Q_n denotes the complex quadric in $\mathbb{C}\mathbb{P}^{n+1}$ which is isomorphic to the real Grassmannian $\mathbb{G}_2^+(\mathbb{R}^{n+2})$ of oriented 2-planes, and $\mathbb{O}\mathbb{P}^2$ is the octonionic projective plane $F_4/Spin_9$.

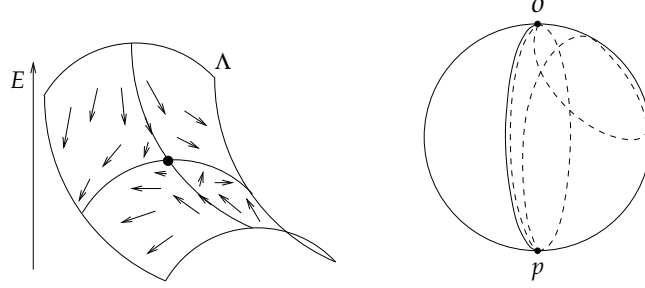
A chain is extendible beyond P_k if and only if P_k contains poles again. E.g. among the Grassmannians $P_3 = \mathbb{G}_p(\mathbb{K}^n)$ only those of half dimensional subspaces ($p = \frac{n}{2}$) enjoy this property: Then (E, E^\perp) is a pair of poles for any $E \in \mathbb{G}_{n/2}(\mathbb{K}^n)$, and the corresponding midpoint set is the group $O_{n/2}, U_{n/2}, Sp_{n/2}$ since its elements are the graphs of orthogonal \mathbb{K} -linear maps $E \rightarrow E^\perp$, see figure below.



2. CENTRIOLES WITH TOPOLOGICAL MEANING

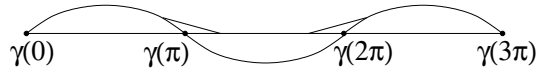
Points in minimal centrioles are in 1:1 correspondence to minimal geodesics between the corresponding poles o and p . Thus minimal centrioles sometimes can be viewed as low-dimensional approximations of the full *path space* Λ , the space of all H^1 -curves⁴ $\lambda : [0, 1] \rightarrow P$ with $\lambda(0) = o$ and $\lambda(1) = p$. This is due to the Morse theory for the energy function E on Λ where $E(\lambda) = \int_0^1 |\lambda'(t)|^2 dt$. We may decrease the energy of any path λ by applying the gradient flow of $-E$ (left figure).

⁴ H^1 means that λ has a derivative almost everywhere which is square integrable. Replacing any path λ by a geodesic polygon with N vertices, we may replace Λ by a finite dimensional manifold, cf. [8].



Most elements of Λ will be flowed to the minima of E which are the shortest geodesics between o and p . The only exceptions are the domains of attraction (“unstable manifolds”) for the other critical points, the non-minimal geodesics between o and p . The codimension of the unstable manifold is the *index* of the critical point, the maximal dimension of any subspace where the second derivative of E (taken at the critical point) is negative. If β denotes the smallest index of all non-minimal critical points, any continuous map $f : X \rightarrow \Lambda$ from a connected cell complex X of dimension $< \beta$ can be moved away from these unstable manifolds and flowed into a connected component of the minimum set, that is into some centriole P_1 . Thus f is homotopic to a map $\tilde{f} : X \rightarrow P_1$.

But this works only if all non-minimal geodesics from o to p have high index ($\geq \beta$). Which symmetric spaces P have this property? An easy example is the sphere, $P = \mathbb{S}^n$. A nonminimal geodesic γ between poles o and p covers a great circle at least one and a half times and can be shortened within any 2-sphere in which it lies (right figure above). There are $n - 1$ such 2-spheres perpendicular to each other since the tangent vector $\gamma'(0) = e_1$ is contained in $n - 1$ perpendicular planes $\text{Span}(e_1, e_i)$ with $i \geq 2$ in the tangent space. Thus the index is $\geq n - 1$, in fact $\geq 2(n - 1)$ since any such geodesic contains at least 2 conjugate points where it can be shortened by cutting the corner, see figure.



For the classical groups we can argue similarly. E.g. in SO_{2n} , a shortest geodesic from I to $-I$ is a product of n half turns, planar rotations by the angle π in n perpendicular 2-planes in \mathbb{R}^{2n} . A non-minimal geodesic must make an additional full turn and thus a 3π -rotation in at least one of these planes, say in the x_1x_2 -plane. This rotation belongs to the rotation group $SO_3 \subset SO_{2n}$ in the $x_1x_2x_k$ -space for any $k \in \{3, \dots, 2n\}$. Using $SO_3 = \mathbb{S}^3/\pm$, we lift the 3π -rotation to \mathbb{S}^3 and obtain a $3/4$ great circle which can be shortened. There are $2n - 2$

coordinates x_k and therefore $2n-2$ independent contracting directions, hence the index of a nonminimal geodesic in SO_{2n} is $\geq 2n-2$ (compare [8, Lemma 24.2]). The index of the spaces P_k can be bounded from below in a similar way, see next section for the chain of SO_n . This implies the homotopy version of the periodicity theorem:

Theorem 2. *When n is even and sufficiently large, we have for $G = SO_{4n}, SU_{2n}, Sp_n$ (notations of table 2):*

$$\pi_k(G) = \pi_{k-1}(P_1) = \pi_{k-2}(P_2) = \pi_{k-3}(P_3) = \pi_{k-4}(P_4).$$

Together with table 2 this implies the following periodicities:

$$\begin{aligned} \pi_{k+2}(SU_n) &= \pi_k(SU_{n/2}), \\ \pi_{k+4}(SO_n) &= \pi_k(Sp_{n/8}), \\ \pi_{k+4}(Sp_n) &= \pi_k(SO_{n/2}). \end{aligned}$$

3. CLIFFORD MODULES

For compact matrix groups G containing $-I$, there is a linear algebra interpretation for the iterated midpoint sets M_j and their components P_j . A geodesic γ in G with $\gamma(0) = I$ is a one-parameter subgroup, and when $\gamma(1) = -I$, then $\gamma(\frac{1}{2}) = J$ is a *complex structure*, $J^2 = -I$. Thus the midpoint set M_1 is the set of complex structures in G . When the connected component P_1 of M_1 contains antipodal points J_1 and $-J_1$, there is a next midpoint set $M_2 \subset P_1$. It consists of points $J_1\gamma(\frac{1}{2})$ where γ is a one-parameter subgroup in G with $\gamma(1) = -I$ such that $J_1\gamma(t)$ is a complex structure for all t ,

$$J_1\gamma J_1\gamma = -I. \quad (*)$$

In particular the midpoint $J = \gamma(\frac{1}{2})$ anticommutes with J_1 (since $J_1 J J_1 J = -I \iff J_1 J = -J J_1$), and when γ is minimal, this condition is sufficient for (*): then both $J_1\gamma J_1$ and $-\gamma^{-1}$ are shortest geodesics from $-I$ to I with midpoint J , so they must agree. By induction hypothesis, we have anticommuting complex structures $J_u \in G$ with $J_i \in P_i$ for $i < k$, and P_k is a connected component of the set

$$(3) \quad M_k = \{J \in G : J^2 = -I, J J_i = -J_i J \text{ for } i < k\}$$

of complex structures $J \in G$ which anticommute with J_1, \dots, J_{k-1} . To finish the induction step we choose some $J_k \in P_k$.

Recall that the *real Clifford algebra* Cl_k is the associative real algebra with 1 which is generated by \mathbb{R}^k with the relations $vw + wv = -2\langle v, w \rangle$. Equivalently, an orthonormal basis e_1, \dots, e_k of $\mathbb{R}^k \subset Cl_k$ satisfies

$$e_i e_j + e_j e_i = -2\delta_{ij}.$$

A *representation* of Cl_k is an algebra homomorphism from Cl_k into some matrix algebra $\mathbb{K}^{n \times n}$ with $\mathbb{K} \in \{\mathbb{R}, \mathbb{C}, \mathbb{H}\}$; the space \mathbb{K}^n on which the matrices operate is called *Clifford module* S . A representation maps the vectors e_i onto matrices J_i with the same relations $J_i^2 = -I$ and $J_i J_j = -J_j J_i$ for $i \neq j$. Thus a Cl_k module is nothing but a *Clifford system*, a family of k are anticommuting complex structures, and the midpoint set $M_{k+1} \subset P_k$ between J_k and $-J_k$ can be viewed as the set of extensions of a given Cl_k -module (defined by J_1, \dots, J_k) to a Cl_{k+1} -module.

The algebraic theory of the Clifford representations is rather easy (cf. [7]). They are direct sums of irreducible representations, and in the real case there is just one irreducible Cl_k -module S_k (up to isomorphisms) when $k \not\equiv 3 \pmod{4}$, while there are two with equal dimensions when $k \equiv 3 \pmod{4}$. For $k = 0, \dots, 8$ we have

Theorem 3.

$$(4) \quad \begin{array}{cccccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ S_k & \mathbb{R} & \mathbb{C} & \mathbb{H} & \mathbb{H} & \mathbb{H}^2 & \mathbb{C}^4 & \mathbb{O} & \mathbb{O} & \mathbb{O}^2 \end{array}$$

and further we have the periodicity theorem for Clifford modules,

$$(5) \quad S_{k+8} = S_k \otimes S_8.$$

For $k = 3$ and $k = 7$, the two different module structures are given by left and right multiplications of $\mathbb{R}^k = \mathbb{K}' := \mathbb{K} \oplus \mathbb{R} \cdot 1$ on $S_k = \mathbb{K}$ for $\mathbb{K} = \mathbb{H}, \mathbb{O}$.

4. INDEX OF NONMINIMAL GEODESICS

From (3) we have gained a uniform description for all iterated centrioles P_k of G in terms of Clifford systems. This can be used for a calculation of the lower bound β for the index of nonminimal geodesics in all P_k , cf. [8].⁵

Theorem 4. *Let $SO_n = G \supset P_1 \supset P_2 \supset \dots \supset P_k \supset \dots$ be the chain (1) of iterated centrioles where n is divisible by a high power of 2. Then for each k there is some lower bound β depending on n such that the index of nonminimal geodesics from J_k to $-J_k$ is $\geq \beta$, and $\beta \rightarrow \infty$ as $n \rightarrow \infty$.*

Proof. Let $\tilde{\gamma} = J_k \gamma : [0, 1] \rightarrow P_k$ be a non-minimal geodesic from J_k to $-J_k$. Then $\gamma(t) = e^{\pi t A}$ for some $A \in \mathfrak{so}_n$. Since $\tilde{\gamma}(t)$ anticommutes with J_i for all $i < k$, it follows that $\gamma(t)$ and A commute with J_i .

⁵A different argument using root systems was given by Bott[(6.7)] [4] and in more detail by Mitchell [9, 10]

Further, from $\tilde{\gamma}(t)^2 = -I$ we obtain $J_k e^{\pi t A} J_k^{-1} = e^{-\pi t A}$ and therefore A anticommutes with J_k . Thus we have computed the tangent space of P_k at J_k :

$$(6) \quad T_{J_k} P_k = \{J_k A : A \in \mathfrak{so}_n, AJ_k = -J_k A, AJ_i = J_i A \text{ for } i < k\}.$$

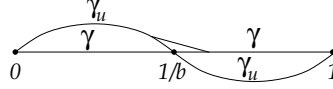
Since $\gamma(1) = -I$, the (complex) eigenvalues of A have the form ai with $i = \sqrt{-1}$ and a an odd integer.

To relate these eigenvalues to the index we argue similar as in [8, p. 144-147]. We split \mathbb{R}^n into a sum of subspaces V_j being invariant under the linear maps A, J_1, \dots, J_k and being minimal with respect to this property. All J_i , $i < k$, preserve the (complex) eigenspaces E_a of A , corresponding to the nonzero eigenvalue ai , while J_k interchanges E_a and E_{-a} . Thus by minimality of V_j , there is just one pair $\pm a$ such that V_j is the real part of $E_a + E_{-a}$. Therefore $J' := A/a$ is an additional complex structure on V_j commuting with J_i ($i < k$) and anticommuteing with J_k , and $J_{k+1} := J_k J'$ is a complex structure which anticommutes with *all* J_1, \dots, J_k . Hence V_j is an irreducible Cl_k -module. Moreover, $A = a_j J'$ on V_j for some nonzero integer a_j while $A = 0$ on V_0 . By choice of the sign of $J'|V_j$ we may assume that all $a_j > 0$. hence $a_j \in \{1, 3, 5, \dots\}$.

Choose two of these irreducible modules, say V_j and V_h . By (4), there is a module isomorphism $V_j \rightarrow V_h$ as Cl_{k+1} -modules when $k+1 \not\equiv 3 \pmod{4}$ (Case 1) and as Cl_k -modules when $k+1 \equiv 3 \pmod{4}$ (Case 2). This remains true when we alter the Cl_{k+1} -module structure of V_h in Case 1 by changing the sign of J_{k+1} (and thus that of J') on V_h . With this identification we have $V_j + V_h = V_j \otimes \mathbb{R}^2$ and $B = I \otimes \begin{pmatrix} & -1 \\ 1 & \end{pmatrix}$ (with $B = 0$ on the other submodules) commutes with all J_j , $j \leq k$, and the same is true for e^{uB} . Putting $A_u = e^{uB} A e^{-uB}$, we have $J_k A_u \in T_{J_k} P_k$ by (6).

Case 1: $k+1 \not\equiv 3 \pmod{4}$: We have modified our identification of V_j and V_h by changing the sign of J_{k+1} on V_h . Thus on $V_j + V_h = V_j \otimes \mathbb{R}^2$ we have $A = J' \otimes D$ where $D = \text{diag}(a_j, -a_h) = cI + D'$ with $D' = \text{diag}(b, -b)$ for $b = \frac{1}{2}(a_j + a_h)$ and $c = \frac{1}{2}(a_j - a_h)$. Let us consider the family of geodesics $J_k \gamma_u$ from J_k to $-J_k$ in P_k with $\gamma_u(t) = e^{t\pi A_u} = e^{uB} \gamma(t) e^{-uB}$. The point $\gamma(t) = e^{\pi t c} e^{\pi t D'}$ is fixed under conjugation with the rotation matrix $e^{uB} = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$ precisely when $e^{\pi t D'} = \text{diag}(e^{\pi t b}, e^{-\pi t b})$ is a multiple of the identity matrix which happens for $t = 1/b$. If one of the eigenvalues a of A is > 1 , say $a_h \geq 3$, then $b = \frac{1}{2}(a_j + a_h) \geq 2$ and $1/b \in (0, 1)$. All γ_u are geodesics connecting I to $-I$ on $[0, 1]$. By “cutting the corner” it follows that γ can no longer be locally shortest beyond $t = 1/b$, see figure. If there is at least one

eigenvalue $a_h > 1$, we have $r - 1$ index pairs (j, h) , hence the index of non-minimal geodesics is at least $r - 1$.



Case 2: $k + 1 \equiv 3 \pmod{4}$: In this case, the product $J_o := J_1 J_2 \dots J_{k-1}$ is a complex structure⁶ which commutes with A and anticommutes with J_k (since $k - 1$ is odd). Thus A can be viewed as a complex matrix, using J_o as the multiplication by i . Let $E_a \subset V_j$ be the eigenspace of A corresponding to the eigenvalue ia where a is any odd integer. Then E_a is invariant under the J_i , $i < k$, which commute with A , but is it also invariant under J_k which anticommutes with A and with $i = J_o$ (since $k - 1$ is odd). By minimality we have $V_j = E_a$, hence $A = aJ_o$. As before, we consider the linear map $B = \begin{pmatrix} & -I \\ I & \end{pmatrix}$ on $V_j + V_h = V_j \otimes \mathbb{R}^2$ and the family of geodesics $\gamma_u(t) = e^{t\pi A_u} = e^{uB}\gamma(t)e^{-uB}$. This time, $A = J' \otimes D$ where $D = \text{diag}(a_j, a_h) = cI + D'$ with $c = \frac{1}{2}(a_j + a_h)$ and $D' = \text{diag}(b, -b)$ with $b = \frac{1}{2}(a_j - a_h)$. Thus the element $\gamma(t) = e^{\pi t c} e^{\pi t D'}$ is fixed under conjugation with the rotation matrix $e^{uB} = \begin{pmatrix} \cos u & -\sin u \\ \sin u & \cos u \end{pmatrix}$ precisely when $e^{\pi t D'} = \text{diag}(e^{\pi t b}, e^{-\pi t b})$ is a multiple of the identity matrix which happens for $t = 1/b$. If $b > 1$, we obtain an energy-decreasing deformation by cutting the corner, see figure above. We need to show that there are enough pairs (j, h) with $b > 1$ when γ is non-minimal.

Any $J \in P_k$ defines a \mathbb{C} -linear map $J_k J$ since $J_k J$ commutes with J_i and hence with J_o . Thus a path $\lambda : I \rightarrow P_k$ from J_k to $-J_k$ defines a family of \mathbb{C} -linear maps, and its complex determinant $\det(J_k \lambda)$ is a path in \mathbb{S}^1 starting and ending at $\det(\pm I) = 1$ (recall that the dimension n is even). This loop in \mathbb{S}^1 has a mapping degree which is apparently invariant under homotopy; it decomposes the path space ΛP_k into infinitely many connected components. If λ is a geodesic, $\lambda(t) = J_k e^{\pi t A}$, then $\det J_k^{-1} \lambda(t) = e^{\pi t \text{trace } A}$, hence its mapping degree is $\frac{1}{2} \text{trace } A/i$. Since $\text{trace } A/i = m \sum_j a_j$, we may fix $c := \sum_j a_j$ (which means fixing the connected component of ΛP_k) and we may assume that $|c|$ is much smaller than r (the number of submodules V_j). Let p denote the sum of the positive a_j and $-q$ the sum of the negative a_j . Then $p + q \geq r$ since all $|a_j| \geq 0$, and $p - q = c$ which means

⁶Putting $S_n = (J_1 \dots J_n)^2$ we have

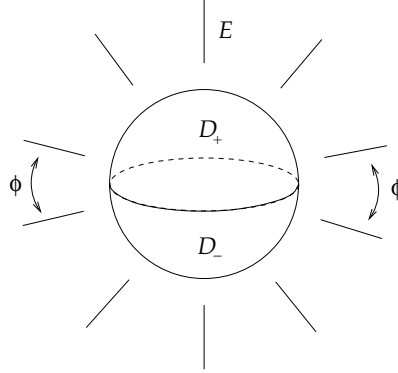
$$S_n = J_1 \dots J_n J_1 \dots J_n = (-1)^{n-1} S_{n-1} J_n^2 = (-1)^n S_{n-1},$$

thus $S_n = (-1)^s I$ with $s = n + (n-1) + \dots + 1 = \frac{1}{2}n(n+1)$. When $n = k-1 \equiv 1 \pmod{4}$, then s is odd, hence $S_n = -I$.

roughly $p \approx q \approx r/2$. Assume for the moment $c = 0$. If there is some eigenvalue a_h with $|a_h| > 1$, say $a_h = -3$, there are many positive a_j with $a_j - a_h \geq 4$, more precisely $\sum_{a_j > 0} (a_j - a_h) \geq 4 \cdot r/2 = 2r$, and this is a lower bound for the index. In the general case this result has to be corrected by the comparably small number c . In contrast, if all $a_j = \pm 1$, the geodesic γ consists of simultaneous half turns in $n/2$ perpendicular planes; these are shortest geodesics from I to $-I$ in SO_n .⁷

5. VECTOR BUNDLES OVER SPHERES

Clifford representations have a direct connection to vector bundles over spheres and hence to K-theory. Every vector bundle $E \rightarrow \mathbb{S}^{k+1}$ is trivial over each of the two closed hemispheres $D_+, D_- \subset \mathbb{S}^{k+1}$, but along the equator $\mathbb{S}^k = D_+ \cap D_-$ the fibres over ∂D_+ and ∂D_- are identified by some map $\phi : \mathbb{S}^k \rightarrow O_n$ called *clutching map*.



Homotopic clutching maps define equivalent vector bundles. Thus vector bundles over \mathbb{S}^{k+1} are classified by the homotopy group $\pi_k(O_n)$. When we allow adding of trivial bundles (stabilization), n may be arbitrarily high. Let \mathcal{V}_k be the set of vector bundles over \mathbb{S}^{k+1} up to equivalence and adding of trivial bundles (“stable vector bundles”). Then

$$(7) \quad \mathcal{V}_k = \lim_{n \rightarrow \infty} \pi_k(O_n).$$

Hence we could apply Theorem 2 in order to classify stable vector bundles over spheres. However, a separate argument based on the same ideas but also using Clifford modules will give more information.

⁷Any one-parameter subgroup γ in SO_n is a family of planar rotations in perpendicular planes. When $\gamma(1) = -I$, all rotation angles are odd multiples of π . The squared length of γ is the sum of the squared rotation angles. Thus the length is minimal if all rotation angles are just $\pm\pi$.

A Cl_k module $S = \mathbb{R}^n$ or the corresponding Clifford system $J_1, \dots, J_k \in O_n$ defines a peculiar map $\phi = \phi_S : \mathbb{S}^k \rightarrow O_n$ which is *linear*, that is a restriction of a linear map $\phi : \mathbb{R}^{k+1} \rightarrow \mathbb{R}^{n \times n}$, where we put

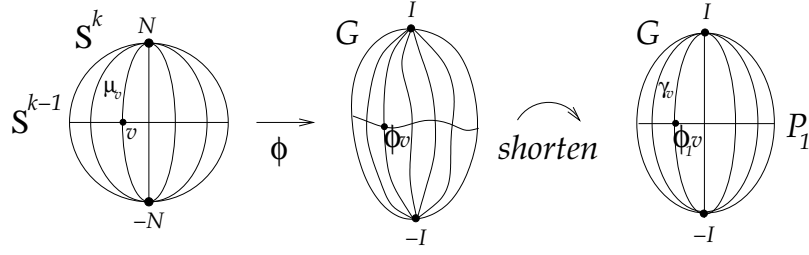
$$(8) \quad \phi_S(e_{k+1}) = I, \quad \phi_S(e_i) = J_i \text{ for } i \leq k.$$

The bundles defined by such clutching maps ϕ_S are called *generalized Hopf bundles*. In the cases $k = 1, 3, 7$, these are the classical complex, quaternionic, and octonionic Hopf bundles over \mathbb{S}^{k+1} .

In fact, Cl_k -modules are in 1:1 correspondence to linear maps $\phi : \mathbb{S}^k \rightarrow O_n$ with the identity matrix in the image. To see this, let ϕ be such map and $W = \phi(\mathbb{R}^{k+1})$ its image. Then $\mathbb{S}_W := \phi(\mathbb{S}^k) \subset O_n$. Thus ϕ is an isometry for the inner product $\langle A, B \rangle = \frac{1}{n} \text{trace}(A^T B)$ on $\mathbb{R}^{n \times n}$ since $\phi(\mathbb{S}^k) \subset O_n$ and O_n lies in the unit sphere of $\mathbb{R}^{n \times n}$. For all $A, B \in \mathbb{S}_W$ we have $(A + B) \in \mathbb{R} \cdot O_n$. On the other hand, $(A + B)^T(A + B) = 2I + A^T B + B^T A$, thus $A^T B + B^T A = tI$ for some $t \in \mathbb{R}$. From the inner product with I we obtain $t = 2\langle A, B \rangle$. Inserting $A = I$ and $B \perp I$ yields $B + B^T = 0$, and for any $A, B \perp I$ we obtain $AB + BA = -2\langle A, B \rangle I$. Thus $\phi|_{\mathbb{R}^k}$ defines a Cl_k -representation on \mathbb{R}^n .

Atiyah, Bott and Shipiro [3] reduced the theory of vector bundles over spheres to the simple algebraic structure of Clifford modules by showing that all vector bundles over spheres are generalized Hopf bundles plus trivial bundles, see Theorem 5 below. We sketch a different proof of this theorem using the original ideas of Bott and Milnor. We will homotopically deform the clutching map $\phi : \mathbb{S}^k \rightarrow G = SO_n$ of the given bundle $E \rightarrow \mathbb{S}^{k+1}$ step by step into a linear map. Since adding of trivial bundles is allowed, we may assume that the rank n of E is divisible by a high power of 2.

We declare $N = e_{k+1}$ to be the “north pole” of \mathbb{S}^k . First we deform ϕ such that $\phi(N) = I$ and $\phi(-N) = -I$. Thus ϕ maps each meridian from N to $-N$ in \mathbb{S}^k onto some path from I to $-I$ in G , an element of ΛG . The meridians μ_v are parametrized by $v \in \mathbb{S}^{k-1}$ where \mathbb{S}^{k-1} is the equator of \mathbb{S}^k . Therefore ϕ can be considered as a map $\phi : \mathbb{S}^{k-1} \rightarrow \Lambda G$. Using the negative gradient flow for the energy function E on the path space ΛG as in section 2 we may shorten all $\phi(\mu_v)$ simultaneously to minimal geodesics from I to $-I$ and obtain a map $\tilde{\phi} : \mathbb{S}^{k-1} \rightarrow \Lambda_o G$ where $\Lambda_o G$ is the set of shortest geodesics from I to $-I$, the minimum set of E on ΛG . Let $m(\gamma) = \gamma(\frac{1}{2})$ be the midpoint of any geodesic $\gamma : [0, 1] \rightarrow G$. Thus we obtain a map $\phi_1 = m \circ \tilde{\phi} : \mathbb{S}^{k-1} \rightarrow P_1$, and we may replace ϕ by the geodesic suspension over ϕ_1 from I and $-I$.

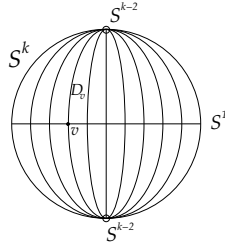


We repeat this step replacing G by P_1 and ϕ by ϕ_1 . Again we choose a “north pole” $N_1 = e_k \in \mathbb{S}^{k-1}$ and deform ϕ_1 such that $\phi_1(\pm N_1) = \pm J_1$ for some $J_1 \in P_1$. Now we deform the curves $\phi_1(\mu_1)$ for all meridians $\mu_1 \subset \mathbb{S}^{k-1}$ to shortest geodesics, whose midpoints define a map $\phi_2 : \mathbb{S}^{k-2} \rightarrow P_1$, and then we replace ϕ_1 by a geodesic suspension from $\pm J_1$ over ϕ_2 . This step is repeated $(k-1)$ -times until we reach a map $\phi_{k-1} : \mathbb{S}^1 \rightarrow P_{k-1}$. This loop can be shortened to a geodesic loop $\tilde{\gamma} = J_{k-1}\gamma : [0, 1] \rightarrow P_{k-1}$ (which is a closed geodesic since P_{k-1} is symmetric) starting and ending at J_{k-1} , such that $\tilde{\gamma}$ and γ are shortest in their homotopy class.

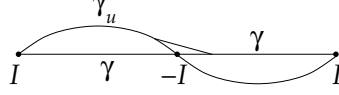
We have $\gamma(t) = e^{2\pi t A}$ for some $A \in T_{J_{k-1}}P_{k-1}$. Since γ is closed, the (complex) eigenvalues of A have the form ai with $a \in \mathbb{Z}$ and $i = \sqrt{-1}$. To compute these eigenvalues we argue as in section 4. We split \mathbb{R}^n into $V_0 = \ker A$ and a sum of subspaces V_j which are invariant under the linear maps A, J_1, \dots, J_{k-1} and minimal with respect to this property. As before, $A = aJ'$ for some nonnegative integer a , and $J_k = J_{k-1}J'$ is a complex structure anticommuting with J_1, \dots, J_{k-1} . Hence V_j is an irreducible Cl_k -module with dimension m_k , see (4), (5).

Since the clutching map of the given vector bundle $E \rightarrow \mathbb{S}^{k+1}$ (after the deformation) is determined by $\gamma, J_1, \dots, J_{k-1}$ which leave all $V_j, j \geq 0$, invariant, the vector bundle splits accordingly as $E = E_0 \oplus \sum_{j>0} E_j$ where E_0 is trivial.⁸

⁸The clutching map $\phi : \mathbb{S}^k \rightarrow SO_n$ splits into components $\phi_j : \mathbb{S}^k \rightarrow SO(V_j)$. The domain \mathbb{S}^k is the union of totally geodesic spherical $(k-1)$ -discs $D_v, v \in \mathbb{S}^1$, centered at v and perpendicular to \mathbb{S}^1 . All D_v have a common boundary \mathbb{S}^{k-2} . Since $\phi_0|_{D_v}$ is constant in v , it is contractible along D_v to a constant map.



We claim that the minimality of γ implies $a_j = 1$ for all j and hence $A = J_k$. In fact, the geodesic variation γ_u of section 4 shows that $|a_j - a_h| < 2$ for all j, h , otherwise we could shorten γ by cutting the corner.



Now suppose that, say, $a_1 \geq 2$. We may assume that $V_0 = \ker A$ contains another copy \tilde{V}_1 of V_1 as a Cl_{k-1} -module: if not, we extend E_0 by the trivial bundle $\mathbb{S}^{k+1} \times \tilde{V}_1$. Thus we have eigenvalues 0 and a_1 on $\tilde{V}_1 \oplus V_1$ with difference ≥ 2 , in contradiction to the minimality of the geodesic.

We have shown $E = E_0 \oplus E_1$ where E_0 is trivial and E_1 is a generalized Hopf bundle for the Clifford system J_1, \dots, J_k on $\sum_{j>0} V_j$.

Let \mathcal{M}_k the set of equivalence classes of Cl_k -modules, modulo trivial Cl_k -representations. We have studied the map

$$\hat{\alpha} : \mathcal{M}_k \rightarrow \mathcal{V}_k$$

which assigns to each $S \in \mathcal{M}_k$ the corresponding generalized Hopf bundle over \mathbb{S}^{k+1} . It is additive with respect to direct sums. We have just proved that $\hat{\alpha}$ is onto. But it is not 1:1. In fact, every Cl_{k+1} -module is also a Cl_k module since $Cl_k \subset Cl_{k+1}$. This defines a restriction map $\rho : \mathcal{M}_{k+1} \rightarrow \mathcal{M}_k$. Any Cl_k -module S which is really a Cl_{k+1} -module gives rise to a contractible clutching map $\phi_S : \mathbb{S}^k \rightarrow SO_n$ and hence to a trivial vector bundle since ϕ_S can be extended to \mathbb{S}^{k+1} and thus contracted over one of the half spheres $D_+, D_- \subset \mathbb{S}^{k+1}$. Thus $\hat{\alpha}$ sends $\rho(\mathcal{M}_{k+1})$ into trivial bundles and hence it descends to an additive map⁹

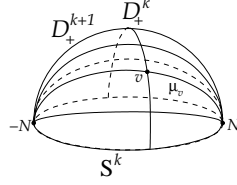
$$\alpha : \mathcal{A}_k := \mathcal{M}_k / \rho(\mathcal{M}_{k+1}) \rightarrow \mathcal{V}_k.$$

We claim that this map is injective: if a bundle $\hat{\alpha}(S)$ is trivial for some Cl_k -module S , then S is (the restriction of) a Cl_{k+1} -module.

Proof of the claim. Let S be a Cl_k -module and $\phi = \phi_S : \mathbb{S}^k \rightarrow G$ the corresponding clutching map (that is $\phi(e_{k+1}) = I$, $\phi(e_i) = J_i$). We assume that ϕ is contractible, that is it extends to $\hat{\phi} : D^{k+1} \rightarrow G$. The closed disk D^{k+1} will be considered as the northern hemisphere $D_+^{k+1} \subset \mathbb{S}^{k+1}$. Repeating the argument above for the surjectivity, we consider the meridians μ_v between $N = e_{k+1} \in \mathbb{S}^k$ and $-N$, but this time there are much more such meridians, not only those in \mathbb{S}^k but also

⁹In fact, both \mathcal{V}_k and \mathcal{A}_k are abelian groups with respect to direct sums, not just semigroups, and α is a group homomorphism. Using the tensor product, $\mathcal{V} = \sum_k \mathcal{V}_k$ and $\mathcal{A} = \sum_k \mathcal{A}_k$ become rings and α a ring homomorphism, see [3].

those through the hemisphere D_+^{k+1} . They are labeled by $v \in D_+^k := D_+^{k+1} \cap N^\perp$.



Applying the negative energy gradient flow we deform the curves $\phi(\mu_v)$ to minimal geodesics without changing those in $\phi(\mathbb{S}^k)$ which are already minimal. Then we obtain the midpoint map $\hat{\phi}_1 : D_+^k \rightarrow P_1$ with $\phi_1(v) = m(\hat{\phi}(\mu_v))$ which extends the given midpoint map ϕ_1 of ϕ . This step is repeated k times until we reach $\hat{\phi}_k : D_+^1 \rightarrow P_k$ which is a path from J_k to $-J_k$ in P_k . This path can be shortened to a minimal geodesic in P_k whose midpoint is a complex structure J_{k+1} anticommuting with J_1, \dots, J_k . Thus we have shown that our Cl_k -module S is extendible to a Cl_{k+1} -module, that is $S \in \rho(\mathcal{M}_{k+1})$. This finishes the proof of the injectivity.

Theorem 5. [3] *Every vector bundle over \mathbb{S}^k splits into a trivial bundle and a generalized Hopf bundle. More precisely, the map $\alpha : \mathcal{A}_k = \mathcal{M}_k / \rho(\mathcal{M}_{k+1}) \rightarrow \mathcal{V}_k$ sending the equivalence class of a Cl_k -module S onto its generalized Hopf bundle is an isomorphism.*

From (4) one easily obtains the groups \mathcal{A}_k since the modules S_k in (4) are the (one or two) generators of \mathcal{M}_k . If $S_k = \rho(S_{k+1})$, then $\mathcal{A}_k = 0$. This happens for $k = 2, 4, 5, 6$. For $k = 0, 1$ we have

$$\rho(S_{k+1}) = S_k \oplus S_k = 2S_k,$$

hence $\mathcal{A}_0 = \mathcal{A}_1 = \mathbb{Z}_2$. For $k = 3, 7$ there are two generators for \mathcal{M}_k , say S_k and S'_k , and $\rho(S_{k+1}) = S_k \oplus S'_k$, thus $\mathcal{A}_3 = \mathcal{A}_7 = \mathbb{Z}$. Hence

$$(9) \quad \begin{array}{c|cccccccc} k & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline \mathcal{A}_k & \mathbb{Z}_2 & \mathbb{Z}_2 & 0 & \mathbb{Z} & 0 & 0 & 0 & \mathbb{Z} \end{array}$$

and because of the periodicity (5) we have $\mathcal{A}_{k+8} = \mathcal{A}_k$.

Consequently, the list (9) for \mathcal{A}_k is the same as that for \mathcal{V}_k and for $\pi_k(O_n)$, n large (see (7)). Thus we have also computed the stable homotopy of O_n .

We have seen that the following objects are closely related and obey the same periodicity theorem:

- Iterated centrioles of O_n ,
- stable homotopy groups of O_n ,

- Clifford modules,
- stable vector bundles over spheres.

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